INTERIOR SCHAUDER ESTIMATES FOR PARABOLIC DIFFERENTIAL- (OR DIFFERENCE-) EQUATIONS VIA THE MAXIMUM PRINCIPLE

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ABSTRACT

A-priori pointwise estimates to difference-quotients of solutions to elliptic or parabolic equations can be obtained by using the maximum property of appropriate higher-dimensional operators. This method, introduced by Brandt, is here used for a simple derivation of the interior Schauder estimates for second-order parabolic differential equations. The same derivation is applicable also for the analogous finite-difference equations.

1. **Introduction.** The Schauder (and similar) estimates play a fundamental role in the existence and regularity theory for linear and non-linear elliptic and parabolic differential equations. The discrete analogue of these estimates is the basic tool in the more general existence-regularity-and-convergence theory for elliptic and parabolic *finite difference* equations. In this paper we present a new proof to the interior Schauder estimates, which is much simpler than former proofs and is applicable to both differential and finite-difference equations.

The tool we use in our proofs is the strong maximum principle, which is described in Section 3. The derivation of this principle (due to E. Hopf in the elliptic case and extended by L. Nirenberg to the parabolic case) is quite simple and is based on just elementary concepts of the calculus. In Section 4 the maximum principle (for a certain higher-dimensional operator) is applied and easily yields the "fundamental result" (Theorem 1), which is a special case of the interior Schauder estimates. The full estimates (Theorem 3) could be established in a similar way, but with much more complex operators and comparison functions. Compare [1], where this was done for the elliptic case. Here, however, we chose to derive the full Schauder estimates by using (Section 5) the fundamental result and a perturbation technique similar to the one used by Douglis and Nirenberg [3].

Received August 18, 1969

Since second-order finite-difference parabolic operators also satisfy a maximum principle (not always in its conventional form, though; cf. [2]), the proof in Section 4 can be easily carried over, line by line, to the discrete analog. The perturbation procedure can also be easily translated to the finite difference case.

Previous deivations of Schauder estimates are based on estimates for a fundamental solution in terms of which the general solution can be represented by integrals. See Friedman [4], Chapter 4, with bibliographical remarks on page 335. Recently V. Thomée [5] developed a similar approach to the discrete case. Our technique is not only simpler, but is also capable of producing stronger results, such as interior estimates for finite-difference equations with *discontinuous coefficients* (cf. [1]), and also *boundary* estimates, again with possibly discontinuous coefficients (cf. forthcoming paper). In this paper we allow the coefficients to be *discontinouus as functions of t*. (See Remark 2 in Section 5).

Our use of the strong maximum principle has, however, the disadvantage of restricting the applicability of our technique to the cases where such a principle is established. No such restriction is made in [5]. In fact, all we really show is that the maximum principle is, in a sense, a deeper concept than apriori derivatives estimates, because it directly implies them (provided we can extend the maximum principle to higher dimensions). The present author does not know of any strong maximum principle which holds for *higher-order* differential (or difference) elliptic or parabolic equations or systems, except for the case where the higher-order elliptic operator can be written as a product of second-order elliptic operators. In this special case our method is applicable. In the general case it should be noted that without a maximum theorem all other apriori estimates are anyhow useless for the convergence theory. (For instance, to use Theorem 3 below, M_0 should be first estimated.)

2. **Preliminary notation.** We shall use the following notation. E_n is the real n-space. E_n^1 is the $n+1$-dimensional space with points $P = (x,t)$, where $x \in E_n$ and $t \in E_1$. We denote by $e_k = (e_{k1}, \cdots, e_{kn}, e_{k,n+1}) \in E_n^1$ the unit vector in the kth direction, that is, $e_{kk} = 1$ and $e_{ki} = 0$ for $i \neq k$. For any function $F(x,t)$ on E_n^1, and for any scalar η and $1 \leqq k \leqq n$ we define

$$\delta_k(\eta)F(P) = \tfrac{1}{2}[F(P+\eta e_k) - F(P-\eta e_k)]$$

$$\mu_k(\eta)F(P) = \tfrac{1}{2}[F(P+\eta e_k) + F(P-\eta e_k)].$$

It is easy to see that

$$\delta_k\{F \cdot G\} = \delta_k F \cdot \mu_k G + \mu_k F \cdot \delta_k G \tag{2.1}$$

For a multi-index $k = (k_1, k_2, k_3)$ and $y = (y_1, y_2, y_3) \in E_3$ we introduce the higher-order difference operator

$$\delta_k^3(y) = \delta_{k_1}(y_1)\delta_{k_2}(y_2)\delta_{k_3}(y_3).$$

3. **The maximum principle.** In this section we shall state the strong maximum principle for a general parabolic differential operator L. Note, however, that in the next sections we only use the maximum principle for the special case where L coincides with its principal part and the coefficients a_{ij} are independent of x. In fact, we only use Corollary 1, which is a weaker form of the maximum principle.

Consider the operator

$$\text{(3.1)} \qquad Lu \equiv \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} - au - \frac{\partial u}{\partial t}$$

in a certain domain $\Omega \subset E_n^1$. Here $a_{ij} = a_{ji} = a_{ij}(x,t)$, $a_i = a_i(x,t)$ and $a = a(x,t) \leqq 0$ are given bounded functions on Ω, while $u = u(x,t)$ is an unknown function. L is assumed to be locally uniformly elliptic, i.e.,

$$\text{(3.2)} \qquad \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geqq v(x,t) \sum_{i=1}^{n} \xi_i^2 \quad \text{for all } \xi \in E_n,\ (x,t) \in \Omega,$$

where $v(x,t)$ is a positive continuous function independent of ξ.

Notation. For any $P^0 = (x^0, t^0) \in \Omega$ we denote by $S(P^0)$ the set of all points Q in Ω which can be connected to P^0 by a simple continuous curve in Ω along which the t-coordinate is non-decreasing from Q to P^0. The set of all points Q on the *boundary* of Ω which can be so connected to P^0 will be denoted $T(P^0)$.

The strong maximum principle. *If $Lu \geqq 0$ in Ω and if u has a positive maximum at $P^0 \in \Omega$ then $u(P) = u(P^0)$ for all $P \in S(P^0)$.*

Proof. See Friedman [4], page 34, with bibliographical notes on page 335.

Corollary 1. *If $L\psi \leqq -|L\phi|$ in a bounded domain Ω on the boundary of which ψ and ϕ are continous, and if $\psi \geqq |\phi|$ on $T(P)$, then $\psi(P) \geqq |\phi(P)|$.*

Proof. By the maximum principle $\phi - \psi$ cannot have a positive maximum in $\overline{S(P)}$ (= the closure of $S(P)$) and therefore $\phi - \psi \leqq 0$ throughout $\overline{S(P)}$. In particular $\psi(P) \geqq \phi(P)$. Similarly $\psi(P) \geqq -\phi(P)$. Q.E.D.

4. **The fundamental result.** Let N be the semicube in E_n^1 with top $(0,0)$ and edge d:

$$N = \{(x,t) \mid |x_i| \leqq d, \ -t_0 \leqq t \leqq 0\}, \quad t_0 = \frac{d^2}{a_0}.$$

Consider the parabolic operator

$$L_0 u = \sum_{i,j=1}^{n} a_{ij}(t) \frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t}, \tag{4.1}$$

where $a_{ij} = a_{ji}$, $\sum a_{ij}(t)\xi_i\xi_j > \nu \sum \xi_i^2$ for all $\xi \in E_n$, and $\sum_{i=1}^{n} a_{ii}(t) \leqq \bar{a}$, ν and $\bar{a}$ being positive constants.

THEOREM 1. *If $L_0 u = f(x,t)$ in N and if f is Hölder Continuous (exponent α), then for any $1 \leqq k \leqq n$ and $0 < \eta \leqq d$*

$$\frac{|\delta_k(\eta) D^2 u(0,0)|}{\eta^\alpha} \leqq C_\alpha \nu^{-1} h_f + (4+K)\eta^{1-\alpha} d^{-1} m_2, \tag{4.2}$$

where D^2 is any second-order x-derivative,

$$h_f = \sup_{k,\eta,x,t} \frac{|\delta_k(\eta) f(x,t)|}{\eta^\alpha}$$

$$m_2 = \sup_{x,t,D^2} |D^2 u(x,t)|$$

$$C_\alpha = 6(1-\alpha^2)^{-1} 3^{1/\alpha}$$

and

$$K = \nu^{-1}(16\bar{a} + \tfrac{1}{2} a_0)$$

Proof. We introduce the $(n+4)$-dimensional domain ($y \in E_3$)

$$N_3 = \left\{(x,t,y) \;\middle|\; |x_j| < \frac{d}{4}, \quad -t_0 < t < 0, \quad 0 < y_l < \frac{d}{4}\right\}$$

on which the function

$$\phi = \phi(x,t,y) = \delta_k^3(y) u(x,t)$$

is well-defined, for any fixed multi-index $k = (k_1, k_2, k_3)$. By the theorem of the mean we easily see that

$$|\phi(x,t,y)| \leqq m_2 y_k y_l, \quad (k,l) = (1,2),\ (2,3) \text{ or } (3,1). \tag{4.3}$$

We also introduce the $(n+4)$-dimensional uniformly parabolic operator

$$L_3 = L_0 + \frac{\nu}{4} \sum_{l=1}^{3} \left(\frac{\partial^2}{\partial y_l^2} - \frac{\partial^2}{\partial x_{k_l}^2}\right)$$

which satisfies

$$L_3\phi(x,t,y) = \delta_k^3(y)L_0u(x,t) = \delta_k^3(y)f(x,t)$$

and therefore

(4.4) $$\left|L_3\phi(x,t,y)\right| \leqq h_f y_*^\alpha, \quad \text{where } y_* = \min(y_1,y_2,y_3).$$

Next we define the *comparison function*

(4.5) $$\psi(x,t,y) = h_f\psi_1 + m_2\psi_2 + m_2\psi_3$$

where

$$\psi_1 = C_\alpha v^{-1}y_1y_2y_3(y_1^\alpha + y_2^\alpha + y_3^\alpha)^{-(1-\alpha)/\alpha}$$

$$\psi_2 = d^{-1}y_1y_2y_3[4 + K(1 - \frac{4}{d}y_1)]$$

$$\psi_3 = y_2y_3\left[\frac{16}{d^2}\sum_{j=1}^n x_j^2 - \frac{t}{t_0}\right].$$

It is clear that $\psi_i \geqq 0$ throughout $\bar{N}_3$ (= the closure of N_3) and in particular

$$\psi \geqq 0 = |\phi| \quad \text{on} \quad B_0 = \bar{N}_3 \cap \bigcup_{l=1}^3 \{y_l = 0\},$$

$$\psi \geqq m_2\psi_2 \geqq \frac{4}{d}m_2y_1y_2y_3 \geqq m_2y_ky_l \quad \text{on} \quad B_1 = \bar{N}_3 \cap \bigcup_{l=1}^3 \left\{y_l = \frac{d}{4}\right\}$$

$$\psi \geqq m_2\psi_3 \geqq m_2y_2y_3 \quad \text{on} \quad B_2 = \bar{N}_3 \cap \left(\{t = -t_0\} \cup \bigcup_{j=1}^n \left\{|x_j| = \frac{d}{4}\right\}\right),$$

and therefore, by (4.3),

(4.6) $$\psi \geqq |\phi| \quad \text{on} \quad B = B_0 \cup B_1 \cup B_2.$$

By direct differentiations we easily find that in N_3

(4.7) $$L_3\psi_3 \leqq d^{-2}y_2y_3(32\bar{a} + a_0) = -L_3\psi_2$$

and

$$L_3\psi_1 \leqq -\tfrac{1}{4}C_\alpha(1-\alpha^2)(y_1^\alpha + y_2^\alpha + y_3^\alpha)^{-(1+\alpha)/\alpha}\sum_{3!} y_k^{\alpha-1}y_ly_m^{\alpha+1},$$

where (k,l,m) is a permutation of $(1,2,3)$ and $\Sigma_{3!}$ indicates summation on all such permutations. Therefore, taking now $y_* = y_k \leqq y_l \leqq y_m$, we have

$$\begin{aligned} L_3\psi_1 &\leqq -\tfrac{1}{4}C_\alpha(1-\alpha^2)(3y_m^\alpha)^{-(1+\alpha)/\alpha}(y_k^{\alpha-1}y_l + y_l^{\alpha-1}y_k)y_m^{\alpha+1} \\ &= -\tfrac{1}{2}(y_k^{\alpha-1}y_l + y_l^{\alpha-1}y_k) \\ &\leqq -y_*^\alpha. \end{aligned}$$

This, together with (4.4), (4.5) and (4.7), entails

(4.8) $$L_3\psi \leqq -|L_3\phi|.$$

By Corollary 1, (4.6) and (4.8) involve $\psi \geqq |\phi|$ throughout N_3. In particular

$$|\delta_k^3(y)u(0,0)| = |\phi(0,0,y)| \leqq \psi(0,0,y) = h_f\psi_1 + m_2\psi_2.$$

Dividing this through by $y_1y_2y_3^{\alpha}$ and letting y_1 and y_2 tend to zero, we get (4.2) for $k = k_3$, $\eta = y_3 \leqq d/4$. For $\eta \geqq d/4$, (4.2) results directly from the definition of m_2. Q.E.D.

5. **Interior Schauder estimates by perturbation.** Let Ω be a bounded domain in E_n^1, and let $S(P)$ and $T(P)$ be defined as in Section 3. The distance between two points $P = (x_1^P, \cdots, x_n^P, t^P)$ and $Q = (x_1^Q, \cdots, x_n^Q, t^Q)$ in E_n^1 is defined by

$$|P-Q| = \max\{|x_1^P - x_1^Q|, \cdots, |x_n^P - x_n^Q|,\ a_0|t^P - t^Q|^{\frac{1}{2}}\}.$$

For $P, Q \in \Omega$ we put

$$d_P = \inf_{R \in T(P)} |P - R|, \qquad d_{PQ} = \min\{d_P, d_Q\}.$$

Considering the uniformly parabolic operator (3.1)—(3.2 with constant ν), we introduce the following bounds:

$$K_0 = \sup d_P^2 a(P), \qquad K_\alpha = \sup d_{PQ}^{2+\alpha} \frac{|a(P) - a(Q)|}{|P-Q|^\alpha},$$

$$K_1 = \sup d_P \sum_i |a_i(P)|, \qquad K_{1+\alpha} = \sup d_{PQ}^{1+\alpha} \sum_i \frac{|a_i(P) - a_i(Q)|}{|P-Q|^\alpha},$$

$$K_2 = \tfrac{1}{2}a_0 + 16 \sup \sum_i a_{ii}(P), \quad K_{2+\alpha} = \sup d_{PQ}^{\alpha} \sum_{i,j} \frac{|a_{ij}(P) - a_{ij}(Q)|}{|P-Q|^\alpha},$$

$$M_q = \sup d_P{}^q |D^q u(P)|, \qquad M_{q+\alpha} = \sup d_{PQ}^{q+\alpha} \frac{|D^q u(P) - D^q u(Q)|}{|P-Q|^\alpha},$$

$$H_f = \sup d_{PQ}^{2+\alpha} \frac{|f(P) - f(Q)|}{|P-Q|^\alpha},$$

where all these sup's are taken over all $P \in \Omega$ and $Q \in \Omega$ such that $P - Q = \eta e_k$ $1 \leqq k \leqq n$, η scalar), and over all q-order derivatives D^q. The exponent α is fixed, $0 < \alpha < 1$.

THEOREM 2. *If $Lu = f$ and if the above bounds are finite, then*

(5.1) $$M_{2+\alpha} \leqq 7C_\alpha\nu^{-1}\overline{KM} + 5\lambda^{-\alpha}(4 + \nu^{-1}K_2)M_2 + 7C_\alpha\nu^{-1}H_f$$

where

$$\overline{KM} = (K_{2+\alpha} + K_1)M_2 + (K_{1+\alpha} + K_0)M_1 + K_\alpha M_0$$

and

$$\lambda = \min\{\tfrac{1}{2}, \tfrac{2}{3}(\tfrac{9}{2} nC_\alpha v^{-1} K_{2+\alpha})^{-1/\alpha}\} \tag{5.2}$$

Proof. Let $\lambda_1 = \lambda/(1+\lambda)$. By definitions, for any $\frac{1}{2} < \theta < 1$, there are two points Q, R and $1 \leqq k \leqq n$ such that $R - Q = \eta e_k$ and $\theta M_{2+\alpha} \leqq v(Q,R)$, where

$$v(Q,R) = d_{QR}^{2+\alpha} \max_{D^2} \frac{|D^2u(Q) - D^2u(R)|}{|Q-R|^\alpha}.$$

Put $P = (Q+R)/2 = Q + \frac{1}{2}\eta e_k$. If $\eta \geqq 2\lambda_1 d_P$ then clearly

$$v(Q,R) \leqq \left(\frac{d_{QR}}{\eta}\right)^\alpha . 2M_2 \leqq \left(\frac{d_P + \eta/2}{\eta}\right)^\alpha . 2M_2 \leqq 2\lambda^{-\alpha} M_2$$

and (5.1) is established. We thus have only to consider the case $\eta < 2\lambda_1 d_P$. Let N be the semicube with top P and edge $d = \lambda_1 d_P$. Then clearly $Q \in N$, $R \in N$. Let

$$L_0 u(x,t) = \sum_{i,j} a_{ij}(x^P, t) \frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t},$$

$$g(x,t) = \sum_{i,j} [a_{ij}(x^P,t) - a_{ij}(x,t)] \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_i a_i(x,t) \frac{\partial u}{\partial x_i} - a(x,t)u + f(x).$$

Applying Theorem 1 to the equation $L_0 u = g$ in N yields

$$v(Q,R) \leqq d_{QR}^{2+\alpha}\{C_\alpha v^{-1} h_g + (4 + v^{-1}K_2)(2d)^{-\alpha} m_2\}$$

where

$$m_2 = \sup_N |D^2 u| \leqq d_1{}^{-2} M_2, \qquad d_1 = \inf_{S \in N} d_S \geqq d_P - d = d/\lambda$$

and

$$h_g = \sup_N \frac{|\delta_k(\zeta) g(x,t)|}{(2\zeta)^\alpha}.$$

For each term of $g(x,t)$ we now use (2.1), and find

$$h_g \leqq d_1^{-2-\alpha}(\lambda^\alpha n K_{2+\alpha} M_{2+\alpha} + \overline{KM} + H_f).$$

Collecting these estimates, together with $d_{QR} \leqq d_P \leqq \frac{3}{2}\lambda^{-1} d$, we obtain

$$\begin{aligned} \theta M_{2+\alpha} &\leqq v(Q,R) \\ &\leqq (\tfrac{3}{2})^{2+\alpha}\{C_\alpha v^{-1}(\lambda^\alpha n K_{2+\alpha} M_{2+\alpha} + \overline{KM} + H_f) + (4 + v^{-1}K_2)(2\lambda)^{-\alpha} M_2\}. \end{aligned}$$

By (5.2) the coefficient of $M_{2+\alpha}$ on the right is not greater than $\frac{1}{2}$. Hence (5.1) follows. Q.E.D.

The following is a simple calculus lemma valid for any function $U \in C^{q+\beta}(\Omega)$.

LEMMA. *If $q > 0$ then for any μ and β, $0 < \mu < 1$, $0 < \beta \leqq 1$,*

$$M_q \leqq (1-\mu)^{-q-\beta}\mu^\beta M_{q+\beta} + (1-\mu)^{-q+1}\mu^{-1}M_{q-1}. \tag{5.3}$$

Proof. By the theorem of the mean, for any $P \in \Omega$, any $1 \leqq k \leqq n$ and any $(q-1)$order x-derivative D^{q-1}, there is a point $P^1 = P + \zeta e_k$ such that $|\zeta| \leqq \mu d_P$ and

$$D_k D^{q-1}u(P^1) = [D^{q-1}u(P+\mu d_P e_k) - D^{q-1}u(P-\mu d_P e_k)](2\mu d_P)^{-1},$$

where $D_k = \partial/\partial x_k$. Substituting this result into the identity

$$D^q u(P) = D_k D^{q-1}u(P^1) - |\zeta|^\beta \frac{D^q u(P^1) - D^q u(P)}{|P^1 - P|^\beta}$$

we get an expression for the q-order derivative in terms of $(q-1)$ derivatives and $(q+\beta)$-order difference-quotients (or derivatives; for $\beta = 1$ the difference quotient in the identity can be replaced by $(q+1)$-order derivative at an intermediate point), from which (5.3) easily follows.

We shall restrict ourselves below to $\mu \leqq 0.01$ only, in which case the lemma yields, for $\gamma = 1/0.99$,

$$M_1 \leqq \gamma^2\bar{\mu}M_2 + \bar{\mu}^{-1}M_0, \qquad M_2 \leqq \gamma^3\mu^\alpha M_{2+\alpha} + \gamma\mu^{-1}M_1.$$

Hence, by taking $\bar{\mu} = \frac{1}{3}\gamma^{-3}\mu$ we easily obtain

$$M_1 \leqq \mu^{1+\alpha}M_{2+\alpha} + 5\mu^{-1}M_0, \qquad (\mu \leqq 0.01) \tag{5.4}$$

$$M_2 \leqq 2\mu^\alpha M_{2+\alpha} + 5\mu^{-2}M_0. \tag{5.5}$$

Taking $\mu = \mu_1$ in (5.4) and $\mu = \mu_2$ in (5.5) and substituting these relations into (5.1) we finally get

THEOREM 3. (*Interior Schauder Estimates*). *If $Lu = f$ and if K_q, $K_{q+\alpha}$, M_q, $M_{q+\alpha}$, $(q = 0,1,2)$ are all finite, then*

$$M_{2+\alpha} \leqq K^* M_0 + 21C_\alpha \nu^{-1}H_f \tag{5.6}$$

where

$$K^* = \tfrac{5}{2}\,\mu_2^{-2-\alpha} + 5\mu_1^{-2-\alpha} + 21C_\alpha\nu^{-1}K_\alpha$$

$$\mu_1 = \min\{[21C_\alpha\nu^{-1}(K_{1+\alpha}+K_0)]^{-1/(1+\alpha)},\ 0.01\}$$

$$\mu_2 = [42C_\alpha\nu^{-1}(K_{2+\alpha}+K_1) + 30\lambda^{-1}(4+\nu^{-1}K_2)]^{-1/\alpha}$$

and C_α and λ are as defined in Theorems 1 and 2.

REMARK 1. By (5.4)–(5.5) this theorem involves estimates also for M_1 and M_2.

REMARK 2. Note that no smoothness restriction is imposed on f and on the coefficients of L *as functions of* t. Such a restriction is necessary only if the smoothness of $\partial u/\partial t$ as a function of t is requested. It is then easy to deduce such a smoothness from similar smoothness of f and the coefficients, by using (5.6) and the relation $Lu = f$.

REMARK 3. Note that the simplicity of our method made it possible for us, without much extra effort, to derive *explicit expressions* for the coefficients in the estimate (5.6). This may be important in applications to numerical-analyses.

REMARK 4. Throughout the above discussion *elliptic equations* can be regarded as the special case $\partial u/\partial t \equiv 0$, and the estimates corresponding to this case are obtained by simply substituting $a_0 = 0$.

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